Calculus  Chapter 1  Limits

Section 1.2 Limits

Limit Facts part 1
1. The answer to a limit is a y-value.
2. The limit tells you to look at a certain x value.
3. If the x value is defined (in the domain), then plug it in to find the y-value

Consider the function $y = x^2 - 2$. At $x = 2$

Limit Facts part 2
1. If the x value is undefined or unreachable, then zoom in from the left and right
2. If the y values are the same on the left and the right, then this is the answer to the limit.
3. If the y values are NOT the same on the left and the right, then the limit is undefined.

Consider the graph of $y = \frac{x^2 + 5x + 6}{x + 2}$ at $x = -2$
Are these limits defined?

Process 1: Make a table that zooms in from left and right

Your Turn:
What is the limit at 2 for the following tables?

<table>
<thead>
<tr>
<th>x</th>
<th>2.1</th>
<th>2.01</th>
<th>2.001</th>
<th>1.9</th>
<th>1.99</th>
<th>1.999</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>4.8</td>
<td>4.93</td>
<td>4.97</td>
<td>5.12</td>
<td>5.06</td>
<td>5.02</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x</th>
<th>1.1</th>
<th>1.01</th>
<th>1.001</th>
<th>1.9</th>
<th>1.99</th>
<th>1.999</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>-2.2</td>
<td>-2.4</td>
<td>-2.45</td>
<td>-2.6</td>
<td>-2.57</td>
<td>-2.52</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x</th>
<th>0.1</th>
<th>0.01</th>
<th>0.001</th>
<th>-0.1</th>
<th>-0.01</th>
<th>-0.001</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>-6</td>
<td>-5.5</td>
<td>-5.1</td>
<td>-4</td>
<td>-4.5</td>
<td>-4.9</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x</th>
<th>8.9</th>
<th>8.99</th>
<th>8.999</th>
<th>9.1</th>
<th>9.01</th>
<th>9.001</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>3.6</td>
<td>3.2</td>
<td>3.14</td>
<td>3</td>
<td>3.04</td>
<td>3.08</td>
</tr>
</tbody>
</table>
Section 1.3 Analyzing Limits

Limit Facts: Properties of limits

**THEOREM 1.2** Properties of Limits

Let $b$ and $c$ be real numbers, let $n$ be a positive integer, and let $f$ and $g$ be functions with the following limits.

\[
\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K
\]

1. Scalar multiple: \(\lim_{x \to c} [bf(x)] = bL\)
2. Sum or difference: \(\lim_{x \to c} [f(x) \pm g(x)] = L \pm K\)
3. Product: \(\lim_{x \to c} [f(x)g(x)] = LK\)
4. Quotient: \(\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K}, \quad \text{provided } K \neq 0\)
5. Power: \(\lim_{x \to c} [f(x)]^n = L^n\)

**Concept Question:** What if the value creates a zero in the denominator or a negative under an even root (This is the same as saying the value is not an element of the domain)?

You Should:

**A Strategy for Finding Limits**

1. Learn to recognize which limits can be evaluated by direct substitution. (These limits are listed in Theorems 1.1 through 1.6.)
2. If the limit of $f(x)$ as $x$ approaches $c$ cannot be evaluated by direct substitution, try to find a function $g$ that agrees with $f$ for all $x$ other than $x = c$. [Choose $g$ such that the limit of $g(x)$ can be evaluated by direct substitution.]
3. Apply Theorem 1.7 to conclude analytically that
   \[
   \lim_{x \to c} f(x) = \lim_{x \to c} g(x) = g(c).
   \]
4. Use a graph or table to reinforce your conclusion.

**Alternate Strategy:**

Plug in a number very close to the value where the limit is being evaluated by hand or on the calculator (trace function).

Example: \(\lim_{x \to 0} \frac{\sqrt[3]{x}}{x}\) you can try to consider .1 and -.1 by hand OR trace to .0001 and -.0001
Day 2 Opener

Investigation Part 1

<table>
<thead>
<tr>
<th>Table 1: limit $x \to 0$ of $\frac{\sin(x)}{x}$</th>
<th>Table 1: limit $x \to 0$ of $\frac{\tan(x)}{x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table 2: limit $x \to 0$ of $\frac{\sin(x)}{x}$</td>
<td>Table 2: limit $x \to 0$ of $\frac{\tan(x)}{x}$</td>
</tr>
<tr>
<td>Table 3: limit $x \to 0$ of $\frac{\sin(x)}{2x}$</td>
<td>Table 3: limit $x \to 0$ of $\frac{\tan(x)}{2x}$</td>
</tr>
<tr>
<td>Table 4: limit $x \to 0$ of $\frac{\sin(x)}{3x}$</td>
<td>Table 4: limit $x \to 0$ of $\frac{\tan(x)}{3x}$</td>
</tr>
<tr>
<td>Table 5: limit $x \to 0$ of $\frac{\sin(3x)}{x}$</td>
<td>Table 5: limit $x \to 0$ of $\frac{\tan(3x)}{x}$</td>
</tr>
<tr>
<td>Table 6: limit $x \to 0$ of $\frac{\sin(2x)}{x}$</td>
<td>Table 6: limit $x \to 0$ of $\frac{\tan(2x)}{x}$</td>
</tr>
<tr>
<td>Table 7: limit $x \to 0$ of $\frac{\sin(4x)}{2x}$</td>
<td>Table 7: limit $x \to 0$ of $\frac{\tan(4x)}{2x}$</td>
</tr>
</tbody>
</table>

Limit Facts: Special Trig. Limits

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$
2. $\lim_{x \to 0} \frac{\tan x}{x} = 1$
3. $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

Special Examples:

Limit as $x \to 0$ of $(\tan x) / 3x$

Limit as $x \to 0$ of $(\sin 4x) / x$
Section 1.4 Continuity

One Sided Limits
If you only want to know what y-value a function approaches as the x-value moves toward 3 (or any other value) from the LEFT, you will see \( \lim_{x \to 3^-} f(x) = \)

If you only want to know what y-value a function approaches as the x-value moves toward 3 (or any other value) from the RIGHT, you will see \( \lim_{x \to 3^+} f(x) = \)

Remember that a limit only exists if the left and right approach the same y-value.

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>Left Limit</th>
<th>Right Limit</th>
<th>Overall Limit and Continuity</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What is continuity at a point?

What does it mean for a function to be everywhere continuous?

What does it mean for a function to be continuous on a closed interval?

The continuity test (3 parts) at a point c:

1. \( f(c) \) is defined
2. \( \lim_{x \to c} f(x) \) exists
3. \( \lim_{x \to c} f(x) = f(c) \)
Section 1.4 Piecewise Functions

If a piecewise function is split at \( x = 1 \), then
- plug 1 into the function describing \( x < 1 \) to find the left hand limit
- plug 1 into the function describing \( x > 1 \) to find the right hand limit
- If the left and right hand limits agree, then the limit as \( x \) approaches 1 exists

Example: \( f(x) = \begin{cases} x & \text{if } x < 1 \\ -x^2 + 5 & \text{if } x \geq 1 \end{cases} \)

Left hand limit as \( x \to 1^- \) plug \( x = 1 \) into \( f(x) = x \) \( f(1) = 1 \)
Right hand limit as \( x \to 1^+ \) plug \( x = 1 \) into \( f(x) = -x^2 + 5 \) \( f(1) = -(1)^2 + 5 = 4 \)

Since the left and right limits do not agree, the limit on the piecewise function does not exist as \( x \to 1 \).

**This means you have to pick up your pencil at \( x = 1 \) as the graph switches from function 1 to function 2.

Section 1.4 Intermediate Value Theorem

**Key Questions:**
What does it mean to say that a function is continuous between 0 and 5?

Draw a graph of a continuous function on the interval 0 to 5 that contains the points \((1, 3)\) and \((4, -1)\) that does NOT have a zero.

**Intermediate Value Theorem (IVT)**
If \( f \) is continuous on the closed interval \([a, b]\) and \( k \) is any number between \( f(a) \) and \( f(b) \), then there is at least one number \( c \) in \([a, b]\) such that \( f(c) = k \).

Simpler:
Consider the interval \([0, 5]\) you considered above.
The two points given were \((1, 3)\) and \((4, -1)\). This can also be written \( f(1) = 3 \) and \( f(4) = -1 \).
Since the \( y \) values are 3 and -1, then all the \( y \) values between 3 and -1 will be defined by some \( x \) value between the \( x \) values of 1 and 4.

Simplest:
Since the point \((1, 3)\) has a positive \( y \)-value and the point \((4, -1)\) has a negative \( y \)-value there has to be a zero (hits the \( x \)-axis) between the \( x \) values of 1 and 4.

**Picture:**
Section 1.5 Infinite Limits

Key Questions:
How can you tell from a table that the y-value approaches infinity (vertical asymptote)?

Consider: \( f(x) = \frac{3}{x+1} \) as \( x \to -1 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( -1.1 )</th>
<th>( -1.01 )</th>
<th>( -1.001 )</th>
<th>Left hand limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>( -0.9 )</th>
<th>( -0.99 )</th>
<th>( -0.999 )</th>
<th>Right hand limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Overall limit:

Picture:

How can you tell from a function that the y-value approaches infinity (vertical asymptote)?

Consider: \( f(x) = \frac{x}{x^2 - x} \)

Parent Function:

Holes:

Vertical Asymptotes:

Picture:
**Key Questions:**
How can you tell from a function that the y-value approaches infinity (vertical asymptote)?

Consider #16: \( h(s) = \frac{(2s - 3)}{(s^2 - 25)} \)

- Parent Function:
- Holes:
- Vertical Asymptotes:
- When is the top zero:
- Table to consider Positives and Negatives:
- Picture:
**Section 3.5 Limits at Infinity & End Behavior**

Unit Circle, quad 2:

**End Behavior**
Limit as \( x \) approaches infinity and negative infinity.

The answer is a \( y \) value.

Both infinity and negative infinity have three possible answers:

1. Infinity
2. Negative Infinity
3. A constant value (This represents a horizontal asymptote.)

**Pictures:**

**Reading the end behavior of rational functions**

**Key Question:** Which gets large the fastest – the top or the bottom?

If the top has a higher degree (gets bigger faster), then the end behavior is infinity or negative infinity. Consider the family of functions.

If the bottom has a higher degree (gets bigger faster), then the end behavior is zero.

If the top and bottom have equal degree (get big at the same rate), then the end behavior is equal to the fraction made by the numbers in front of the highest degree terms from top and bottom.

**Note:** An alternate way to do this is to plug in a large positive number (like 20) to see the end behavior as \( x \) approaches infinity and a large negative number (like -20) to see the end behavior as \( x \) approaches negative infinity. This can be modeled on the calculator using tables or the trace feature.